Efficient Propagation of Error Through System Models for Functions Common in Engineering

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System modeling can help designers make and verify design decisions early in the design process if the model's accuracy can be determined. The formula typically used to analytically propagate error is based on a first-order Taylor series expansion. Consequently, this formula can be wrong by one or more orders of magnitude for nonlinear systems. Clearly, adding higher-order terms increases the accuracy of the approximation but it also requires higher computational cost. This paper shows that truncation error can be reduced and accuracy increased without additional computational cost by applying a predictable correction factor to lower-order approximations. The efficiency of this method is demonstrated in the kinematic model of a flapping wing. While Taylor series error propagation is typically applicable only to closed-form equations, the procedure followed in this paper may be used with other types of models, provided that model outputs can be determined from model inputs, derivatives can be calculated, and truncation error is predictable. [DOI: 10.1115/1.4005444]

1 Introduction

In the system design process, designers frequently experience significant uncertainty in predicting whether a proposed design will meet the design objectives. This is a limiting obstacle in system design [1]. However, if a system model can be obtained and its accuracy quantified, the designer can then verify design decisions early in the design process. This greatly reduces the risk of creating a failed system design. Unfortunately, determining system model accuracy is a significant challenge.

Nondeterministic uncertainty analysis methods can enable the system designer to obtain a statistical output distribution, which often is more meaningful than a simple max/min error bound [2]. Nondeterministic uncertainty analysis methods fall into two main categories: (1) reliability-based design methods [3–5] and (2) methods based on robust design [6–11].

Most nondeterministic error analysis techniques use probabilistic methods to represent sources of uncertainty and then propagate these error sources through a deterministic model [12]. This is commonly done with a Monte Carlo simulation, though other quasi Monte Carlo simulation techniques including Halton [13], Hammersley [14], and Latin supercube sampling [15] have been suggested. A hybrid approach combining a derivative-based method with a Monte Carlo simulation has also been proposed [16]. Some nondeterministic error propagation methods use sub-optimizations [17], others use response surface methodologies [18].

The formula to analytically propagate error through a closed-form equation that is most-often cited in literature is based on a first-order Taylor series [19]. This approach makes several important assumptions and may produce results that are wrong by one or more orders of magnitude for nonlinear functions [20]. While higher-order terms clearly improve the accuracy of the approximation, they also require greater computational cost. This paper presents the derivations of the first- and second-order Taylor series error propagation formulas, discusses their assumptions and limitations, and then shows that the truncation error in lower-order approximations can be predicted and accounted for. Consequently, higher-order accuracy in error propagation can be obtained with lower-order computational cost.

While Taylor series error propagation is typically applicable only to closed-form equations, the procedure followed in this paper may be used with other types of models, provided that model outputs can be determined from model inputs, derivatives can be calculated, and truncation error is predictable.

2 Error Propagation via First-Order Taylor Series

Given a system model, a distribution in input values generally propagates through the model to produce a distribution in outputs. The analytical formula most-often cited in literature that calculates variance propagation is based on a first-order Taylor series expansion. It makes several important assumptions that are limiting in many practical situations. The authors have found that the complete derivation of this formula is absent in textbooks and archival journal literature, though it is fundamental to understanding the limitations of the method and how to overcome them. Consequently, this section presents the derivation of the first-order Taylor series approximation of error propagation in order to more fully describe all of its assumptions and limitations.

2.1 First-Order Formula Derivation. Let y be some function of n inputs x_i. The first-order Taylor series approximation expanded about the input means, $\bar{x}_i$, is shown in Eq. (1)

$$y \approx y_1 = f(\bar{x}_1, \ldots, \bar{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_i - \bar{x}_i) \tag{1}$$

where the partial derivatives are evaluated at the mean $x_i = \bar{x}_i$. An approximation of the output mean $\bar{y}$ is given in Eq. (2)

$$\bar{y}_1 = E[y_1] = f(\bar{x}_1, \ldots, \bar{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mu_{1,i} \tag{2}$$

where E is the expectation operator and $\mu_{k,i}$ is the kth moment for the ith input, $\mu_{k,i} = E[(x_i - \bar{x}_i)^k]$. Subtracting Eq. (1) from Eq. (2) produces Eq. (3)

$$y_1 - \bar{y}_1 = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_i - \bar{x}_i) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mu_{1,i} \tag{3}$$

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Squaring and taking the expectation of Eq. (3) produces Eq. (4)

$$\sigma_y^2 \approx \sum_{i=1}^{n}\left[\left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 - N + \text{COV}\right] \tag{4}$$

where σ_y^2 and σ_x^2 are the variances (second moments) in y and x, respectively. Variance σ^2 is defined in Eq. (5), the non-Gaussian terms N are given by Eq. (6), and the covariance terms COV are given by Eq. (7)

$$\sigma_x^2 = E[(x-\bar{x})^2] = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \tag{5}$$

$$N = \left(\frac{\partial f}{\partial x_i}\right)^2 (\mu_{1,i})^2 + 2\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(\mu_{1,i})(\mu_{1,j}) \tag{6}$$

$$\text{COV} = 2\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\sigma_{x_i x_j}^2 \tag{7}$$

where σ_{xixj}^2 is the covariance between x_i and x_j.

In the case where every input x_i is Gaussian, the first moments ($\mu_{1,i}$) are zero and Eq. (4) reduces to Eq. (8)

$$\sigma_y^2 \approx \sum_{i=1}^{n}\left[\left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 + 2\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\sigma_{x_i x_j}^2\right] \tag{8}$$

When the inputs are also independent, the covariance terms are all zero and Eq. (8) further reduces to Eq. (9)

$$\sigma_y^2 \approx \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 \tag{9}$$

Equation (9) is the formula typically given in literature for analytical error propagation through a system [20–24].

2.2 Summary of Assumptions and Limitations

(1) Error propagation via Taylor series generally assumes the system model y can be represented as a closed-form, differentiable, mathematical equation. However, a closed-form model is not necessary if model outputs can be obtained from given inputs and derivatives can be obtained numerically or analytically.
(2) Taking the Taylor series expansion about a single point ($\bar{x}$) causes the approximation to be of local validity only [16,21]. Consequently, the accuracy of the approximation generally decreases with an increase in the input variance σ_x^2.
(3) The approximation is generally more accurate for linear and polynomial-type models.
(4) All inputs x_i are assumed be Gaussian. When inputs are not Gaussian, the non-Gaussian terms in Eq. (4) given by Eq. (6) cannot be neglected.
(5) All inputs x_i are assumed to be independent. When inputs are not independent, the covariance terms in Eq. (4) cannot be neglected [20,22,25].
(6) The input means and variances must be known.
(7) The output error distribution is assumed to be Gaussian, described by only a mean and standard deviation. A method to achieve a fully-described non-Gaussian output distribution using a Taylor series is currently being researched by the authors [26].

2.3 First-Order Accuracy. Clearly, Eq. (9) is an approximation only and can be wrong by one or more orders of magnitude.

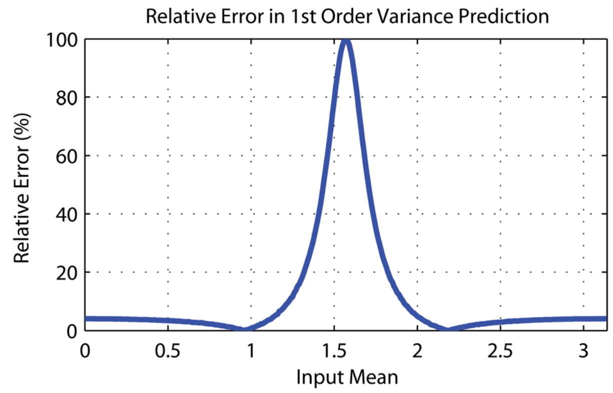

Fig. 1 Relative error in output variance using a first-order Taylor series expansion for the function $y = 1000\sin(x)$

This is especially evident when dealing with nonlinear functions [20].

For example, let y be modeled by the function $y = 1000\sin(x)$. An estimation of the output variance σ_y^2 obtained from Eq. (9) is given in Eq. (10)

$$\sigma_y^2 \approx 10^6 \cos^2(\bar{x})\sigma_x^2 \tag{10}$$

"Actual" output variance was determined from a Monte Carlo simulation using 1 million input values normally distributed about $\bar{x}$ with a standard deviation of $\sigma_x = 0.2$. The relative error ε of Eq. (10) can be calculated using Eq. (11) and the result is plotted as a function of input mean $\bar{x}$ in Fig. 1

$$\varepsilon = \frac{|\sigma_{y,\,predicted}^2 - \sigma_{y,Monte\ Carlo}^2|}{\sigma_{y,Monte\ Carlo}^2} \tag{11}$$

As illustrated in Fig. 1, the first-order approximation of variance propagation is fairly accurate for most values of $\bar{x}$. However, for certain input values, the approximation can be wrong by one or more orders of magnitude, as indicated by the 100% jump in relative error at $\bar{x} = \frac{\pi}{2}$. This spike in relative error occurs because the sin function is nonlinear and the higher-order terms in the Taylor series used to derive Eq. (9) were neglected.

3 Higher-Order Terms

As expected, the accuracy of this estimate of variance propagation through a system can be improved by including higher-order terms in the Taylor series. The derivation of the second-order approximation is presented in this section. The effect of adding the second- and higher-order terms is also discussed.

3.1 Second-Order Formula Derivation. For the sake of brevity, the second-order derivation will be presented only for a monovariable function, $y = f(x)$, though extending this derivation to multivariate functions is trivial. The second-order Taylor series expansion taken about the input mean $\bar{x}$ is given in Eq. (12), and the second-order approximation of the output mean $\bar{y}$ is given in Eq. (13)

$$y \approx y_2 = f(\bar{x}) + \frac{\partial f}{\partial x}(x-\bar{x}) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x-\bar{x})^2 \tag{12}$$

where the partial derivatives are again evaluated at the mean, $x = \bar{x}$

$$\bar{y}_2 = E[y_2] = f(\bar{x}) + \frac{\partial f}{\partial x}\mu_1 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mu_2 \tag{13}$$

The expectation of $(y_2 - \bar{y}_2)^2$ is given in Eq. (14)

$$\sigma_y^2 \approx \left(\frac{\partial f}{\partial x}\right)^2 (\mu_2 - \mu_1^2) + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}(\mu_3 - \mu_1\mu_2) + \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 (\mu_4 - \mu_2^2) \quad (14)$$

If x is Gaussian, all odd moments (μ_k where k is odd) are zero and Eq. (14) reduces to Eq. (15)

$$\sigma_y^2 \approx \left(\frac{\partial f}{\partial x}\right)^2 \mu_2 + \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 (\mu_4 - \mu_2^2) \quad (15)$$

The first-order approximation in Eqs. (8) and (9) requires that only common statistical properties of the input population be known (i.e., mean and variance), but the second-order approximation in Eq. (15) requires fourth central moments. However, for perfectly Gaussian inputs, $\mu_4 \approx 3\sigma^4$ [27]. This substitution is made in Eq. (16)

$$\sigma_y^2 \approx \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 \sigma_x^4 \quad (16)$$

If y is a function of multiple independent inputs, the generalized form of Eq. (16) is given in Eq. (17)

$$\sigma_y^2 \approx \sum_{i=1}^{n}\left(\frac{\partial y}{\partial x_i}\right)^2 \sigma_{x_i}^2 + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial^2 y}{\partial x_i \partial x_j}\right)^2 \sigma_{x_i}^2 \sigma_{x_j}^2 \quad (17)$$

This is the second-order variance propagation formula most-often cited in literature [21,24,28]. Once again, Eqs. (16) and (17) assume all inputs are Gaussian and independent. When this is not the case, the non-Gaussian and covariance terms cannot be neglected.

3.2 Accuracy and Computational Cost. Continuing with the same function $y = 1000\sin(x)$, the relative error obtained from the second-order approximation in Eq. (14) is compared with the relative errors from other-order approximations in Fig. 2.

The second-order approximation successfully filters the large spikes in relative error present in the first-order approximation. However, the second-order approximation still overestimates the actual variance propagation, which could degrade system performance and lead to failure or infeasibility [29]. This bias (about 4%, in this case) is a result of truncating the higher-order terms in the Taylor series.

Figure 2 also illustrates that higher-order terms reduce the second-order bias. With an infinite number of terms, the Taylor series approximation eventually converges to zero error. However, computational cost grows exponentially as higher-order terms are included. This growth in computational cost is accelerated at an exponential rate with an increase in the number of system inputs, as shown in Table 1. Furthermore, higher-order terms also require the calculation of higher-order moments and covariance terms for the system inputs. This exponential growth in cost causes higher-order terms to quickly become prohibitively expensive for complex systems.

Table 1 The number of different order partial derivatives required to propagate variance through a system of various input quantities

Inputs	O1	O2	O3	O4	O5	O6	O7
2	2	3	6	10	15	21	28
4	4	10	20	35	56	84	120
6	6	21	56	126	252	462	792
8	8	36	120	330	792	1716	3138
10	10	55	220	715	2002	5005	11146

4 Higher-Order Accuracy With Lower-Order Cost

This section presents a method to achieve greater accuracy in error propagation through nonlinear systems without increasing computational cost. This is accomplished by predicting the higher-order truncation error and applying a resultant correction factor to the lower-order Taylor series approximation.

This section presents the empirically-determined correction factors for trigonometric, logarithmic, and exponential functions. The procedure presented in this section may be used to determine correction factors for other types of models provided that (1) model outputs can be determined for given inputs, (2) derivatives can be numerically or analytically obtained, and (3) truncation error for that specific model is predictable.

4.1 Reducing Truncation Error. As Fig. 2 illustrates, the second-order truncation error for a sin function is essentially a constant bias for all $\bar{x}$. Figure 3 shows that the magnitude of this bias has a linear relationship to the input variance, σ_x^2. Consequently, this truncation error can easily be estimated and the resultant correction factor e, given in Eq. (18), can be applied to the second-order approximation with Eq. (19)

$$e = \frac{1}{1 + 1.022\sigma_x^2} \quad (18)$$

$$\sigma_{y,CF}^2 = \sigma_y^2(e) \quad (19)$$

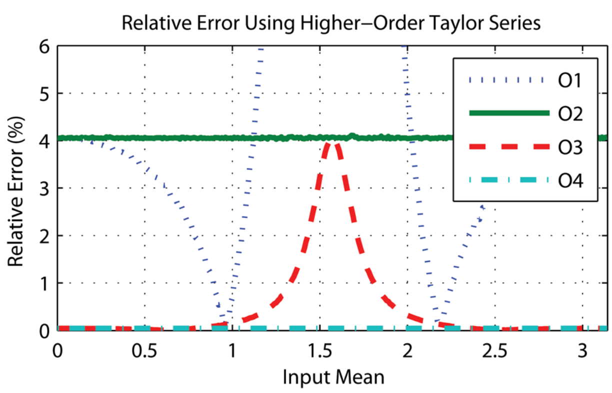

Fig. 2 Relative error in variance propagation using Taylor series approximations

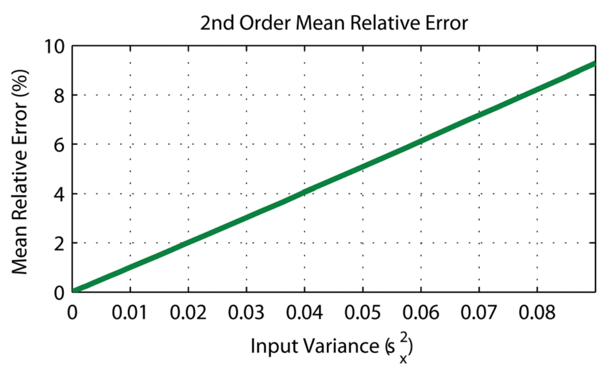

Fig. 3 Relationship between the second-order bias and the input standard variance σ_x^2 for a sin function

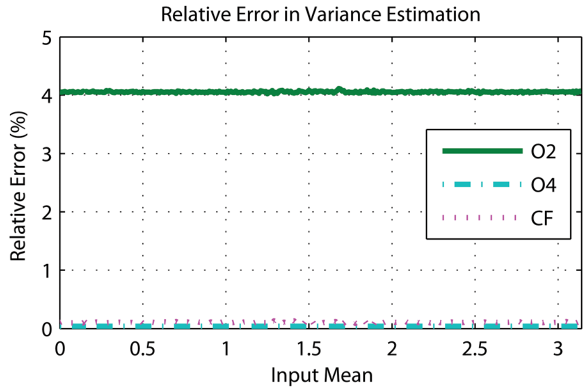

Fig. 4 Relative error in estimations of variance propagation

Table 2 Correction factors to account for higher-order truncation error when propagating error through nonlinear functions with a Taylor series

Func.	Ord.	Correction factor (e)
$y = \sin(x)$	2nd	Eq. (18)
$y = \cos(x)$	2nd	Eq. (18)
$y = \ln(x)$	1st	Eq. (20)
$y = e^x$	1st	Eq. (21)
$y = b^x$	1st	Eq. (22)

The correction factor in Eq. (18) applies to all sin and cos functions. For $y = 1000\sin(x)$, this correction factor and Eq. (19) estimate error propagation with nearly fourth-order accuracy at only second-order computational cost, as higher-order derivatives and input moments are not required.

Figure 4 compares the relative errors in variance estimations. Note that the fourth-order and corrected second-order (CF) approximations are almost identical in Fig. 4.

4.2 Correction Factors for Other Nonlinear Functions. Table 2 gives the correction factors for some common nonlinear functions. Note that the cyclical nature of trigonometric functions and derivatives require a second-order approximation before the higher-order truncation error can easily be determined but the exponential and logarithmic functions only require a first-order calculation. The correction factors for $y = \ln(x)$, $y = \exp(x)$, and $y = b^x$ are given in Eqs. (20)–(22), respectively

$$e = \begin{cases} \exp(-1.9772r + 0.9128) + 1 & \text{if } r >= 0 \\ -3.88r^3 - 4.9835r^2 - 1.5704r + 1.3302 & \text{if } r < 0 \end{cases} \tag{20}$$

where $r = \ln(\frac{\bar{x}}{\sigma_x})$.

$$e = \max\left(1,\ \min\left[2,\ 0.3375\sigma_x^2 + 0.4937\sigma_x + 0.959\right]\right) \tag{21}$$

where e is constrained to a maximum value of 2 and a minimum value of 1

$$e = \begin{cases} \max\left(1,\ \min\left[2,\ \mathbf{X}\cdot(\mathbf{Z_1}\cdot\mathbf{B})^T\right]\right) + 1 & \text{if } b < 1 \\ \max\left(1,\ \min\left[2,\ \mathbf{X}\cdot(\mathbf{Z_2}\cdot\mathbf{B})^T\right]\right) + 1 & \text{if } b > 1 \end{cases} \tag{22}$$

where e is constrained to maximum value of 3 and a minimum value of 2, and

$$\mathbf{X} = \begin{bmatrix} 1 & \sigma_x & \sigma_x^2 & \sigma_x^3 & \sigma_x^4 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & b & b^2 & b^3 \end{bmatrix}^{\mathrm{T}}$$

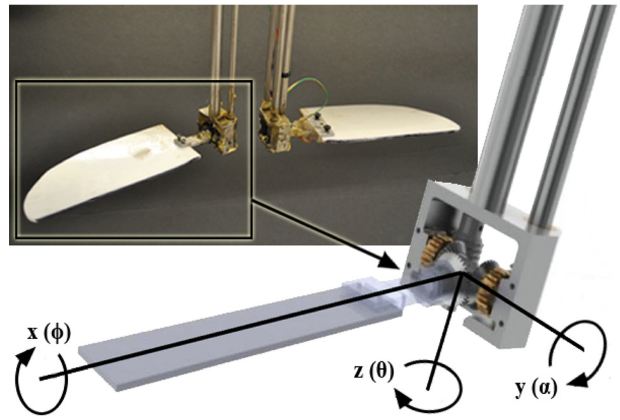

Fig. 5 Mechanism used by the BYU Flapping Flight Research Team to simulate 3-degree-of-freedom motion of a flapping wing

Table 3 Mean values and standard deviations of 16 model input parameters

	Mean	StdDev		Mean	StdDev
$A_{\phi 0}$	−20	0.1	—	—	—
$A_{\phi 1}$	−4	1.5	$B_{\phi 1}$	44	1.5
$A_{\phi 2}$	8	3.0	$B_{\phi 2}$	33	8.0
$A_{\theta 0}$	0	0.1	—	—	—
$A_{\theta 1}$	43	0.75	$B_{\theta 1}$	0	0.1
$A_{\theta 2}$	17	0.5	$B_{\theta 2}$	0	0.1
$A_{\alpha 0}$	12	4.0	—	—	—
$A_{\alpha 1}$	0	0.1	$B_{\alpha 1}$	50	0.1
$A_{\alpha 2}$	−2	0.75	$B_{\alpha 2}$	0	0.1
ω	0.298	0.1			

$$\mathbf{Z_1} = \begin{bmatrix} 0.224 & 0.012 & -2.570e3 & 0 & 0 \\ -0.992 & 0.076 & 1.544e4 & 0 & 0 \\ -0.037 & 7.803 & -2.061e4 & 7.135 & -38.093 \\ 0 & 0 & 0 & 23.556 & 61.532 \end{bmatrix}$$

$$\mathbf{Z_2} = \frac{1}{1000}\begin{bmatrix} -8.612 & 0.018 & -897.1 & 0.006 & 0.005 \\ 1.182 & 0.084 & 986.2 & -0.002 & -0.008 \\ 0.101 & -6.891 & 54.93 & -240.9 & 26.52 \\ -0.002 & 0.089 & -0.276 & -1.743 & 11.06 \end{bmatrix}$$

4.3 Model Composition. It should be noted that the correction factors given in Table 2 are only pertinent to a particular function. If a system model contains this function along with other operators, the system should be decomposed into components (with $\sin(x)$ being a single component, for example). The error should then be propagated through each component individually. The variances in each component's output can then be propagated through the rest of the system model.

5 Example: Kinematic Model of Flapping Wing

Consider the flapping flight wing mechanism shown in Fig. 5. The kinematic model used in the design and optimization of this mechanism is the Fourier series in Eq. (23) [30,31].

$$\begin{bmatrix} \phi(t) \\ \theta(t) \\ \alpha(t) \end{bmatrix} = \sum_{n=0}^{2} \begin{bmatrix} A_{\phi n} \\ A_{\theta n} \\ A_{\alpha n} \end{bmatrix} \cos(n\omega t) + \begin{bmatrix} B_{\phi n} \\ B_{\theta n} \\ B_{\alpha n} \end{bmatrix} \sin(n\omega t) \tag{23}$$

where ϕ is the positional angle (deg), θ is the elevation angle (deg), α is the feathering or attack angle (deg), the As and Bs are Fourier series coefficients (deg), ω is the flapping frequency (hz), and t is time (s). This three output system model has 16 Gaussian inputs (time does not vary), which are statistically described in Table 3 [31].

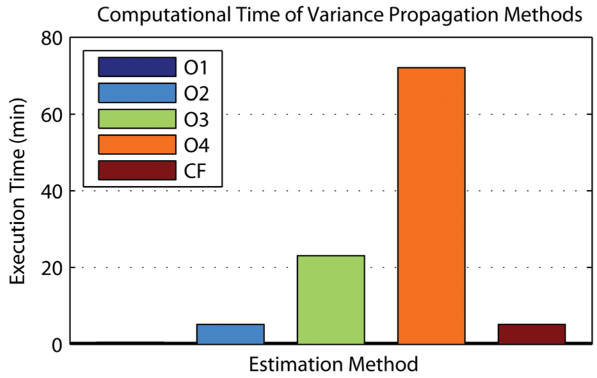

Fig. 6 Computational time to predict output distributions using various error propagation methods

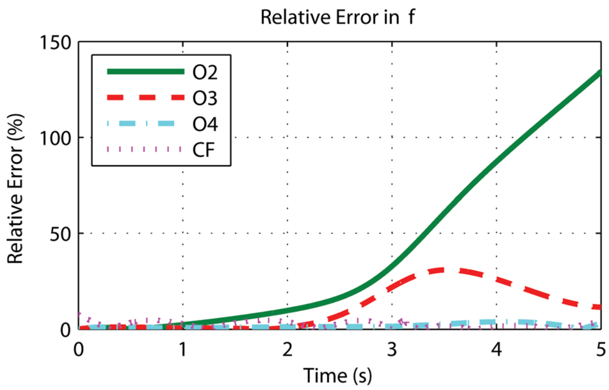

Fig. 7 Relative error in predictions of output variance obtained from various orders of a Taylor series

Various orders of a Taylor series were used to estimate the variance in the three output wing angles based on the variance in these system inputs. The trigonometric correction factors given in Table 2 were then applied to the second-order Taylor series using Eq. (19). Figure 6 shows the computational cost of error propagation (in minutes) using these different methods. These costs are summed over the time interval $0 \text{ s} \le t \le 5 \text{ s}$ at time steps of 0.001 s for all three output angles. The cost of calculating input moments and input covariances from 10k input samples is also included. Computational cost was determined using MATLAB execution time. Notice that the fourth-order prediction took approximately 70 min to execute, where the corrected second-order approximation executed in approximately 4 min—a significant reduction in computational cost.

The relative error in only one of these output angles, ϕ, is shown in Fig. 7, as the other two output angles have similar results. The root-mean-square of the relative error over the time interval shown ($0 \text{ s} \le t \le 5 \text{ s}$) for the second-order approximation is 40.97%, third-order is 11.18%, fourth-order is 1.32%, and a second-order with a correction factor is 1.96%.

Figures 6 and 7 illustrate that the correction factors presented in this paper can achieve near fourth-order accuracy in error propagation through this model with near second-order computational cost.

6 Conclusion

Using a first-order Taylor series to estimate error propagation through a closed-form model is common in literature and practice, but this approximation is frequently used without a full appreciation of its limitations and underlying assumptions. This results in estimations that may be substantially inaccurate. As the authors have been unable to locate the full derivation of this formula either in textbook or archival journal literature, a useful contribution of this paper is the presentation of that derivation and the discussion of the formula's limitations and assumptions.

Additionally, the novel contribution of this paper is the introduction of generic correction factors that account for some of the Taylor series truncation error. These correction factors are predictable, easy to calculate and do not require significant computational cost. They enable a system designer to predict error propagation with greater accuracy, verify design decisions earlier in the design process, and reduce the risk of developing a design that does not meet its objectives. Future research may focus on the development of predictable correction factors for other nonlinear models, such as differential equations and state-space models. The authors believe this can be accomplished using methods based on the same principles discussed in this paper.

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